

Bell inequality, Bell states and maximally entangled states for n qubits

N. Gisin and H. Bechmann-Pasquinucci

Group of Applied Physics, University of Geneva, 1211 Geneva 4, Switzerland

February 1, 2008

Abstract

First, we present a Bell type inequality for n qubits, assuming that m out of the n qubits are independent. Quantum mechanics violates this inequality by a ratio that increases exponentially with m . Hence an experiment on n qubits violating of this inequality sets a lower bound on the number m of entangled qubits. Next, we propose a definition of maximally entangled states of n qubits. For this purpose we study 5 different criteria. Four of these criteria are found compatible. For any number n of qubits, they determine an orthogonal basis consisting of maximally entangled states generalizing the Bell states.

1 Introduction

We consider n qubits, all spatially separated from each other. The purpose of this paper is to find characterizations of maximally entangled states of the n qubits. We assume that the n qubits are equivalent (there is no privileged qubit(s)), hence we restrict ourself to symmetric states. We test the following 5 criteria that could, a priori, characterize such Symmetric Maximally Entangled (SME) states:

1. maximally entangled states violate Bell inequality [1] maximally,
2. maximally entangled states are maximally fragile,
3. whenever $m \leq n$ qubits are measured (in some appropriate basis), the outcomes determine a maximally entangled state of the $m - n$ remaining qubits,
4. the mutual information of measurement outcomes is maximal,
5. all partial traces of maximally entangled states are maximally mixed.

The Bell inequality we use is presented in the next section and the above criteria are made more precise and analysed in section 3. The result of our analysis is that the last one of the above criteria is not a good characterization. However, we find that all the other 4 properties are compatible and define SME states. By local transformation, the SME states generate a basis of the entire Hilbert space.

2 Bell-Klyshko inequality for n qubits

In this section we briefly present a generalization to n qubits of the Bell-CHSH [2] inequality well known in the 2 qubits case. This generalization was first presented by N.D. Klyshko and A.V. Belinskii [3].

Let us start with a brief review of the well known 2 qubits case. Let $a = \pm 1$ and $a' = \pm 1$ denote 2 possible outcomes of 2 possible measurements on the first qubit and similarly $b = \pm 1$ and $b' = \pm 1$ for the second qubit. Let us consider the following linear combination of joint results:

$$F_2 \equiv ab + ab' + a'b - a'b' = (a + a')b + (a - a')b' \leq 2 \quad (1)$$

The above inequality holds because either $a = a'$ and the second term vanishes or $a = -a'$ and the first term vanishes. In the first case the inequality follows because $|a + a'| \leq 2$ and $|b| \leq 1$, and similarly in the second case. Assuming "local realism", the a's and b's are independent and the probabilities of joint results are simply means of products. Using the linearity of the mean operator M and denoting $E_{indep.}(a, b) = M(a.b)$ the expectation value of the products of the results of the experiments a and b , one obtains the traditionally Bell-CHSH inequality [2]:

$$E_{indep.}(a, b) + E_{indep.}(a, b') + E_{indep.}(a', b) - E_{indep.}(a', b') \leq 2 \quad (2)$$

The quantum mechanical description associates to the first measurement the Pauli matrix $\vec{a}\vec{\sigma}$ with normalized 3-dim vectors \vec{a} , and similarly for the other measurements. The mean correlation in state ψ is given by the quantum expectations: $E_{QM}(a, b) = \langle \vec{a}\vec{\sigma} \otimes \vec{b}\vec{\sigma} \rangle_\psi$. Hence, the Bell-CHSH inequality involves the Bell operator:

$$\mathcal{B}_2 = \vec{a}\vec{\sigma} \otimes \vec{b}\vec{\sigma} + \vec{a}\vec{\sigma} \otimes \vec{b}'\vec{\sigma} + \vec{a}'\vec{\sigma} \otimes \vec{b}\vec{\sigma} - \vec{a}'\vec{\sigma} \otimes \vec{b}'\vec{\sigma} \quad (3)$$

A straightforward computation shows that $\mathcal{B}_2^2 \leq 8$. Accordingly, the largest eigenvalue of \mathcal{B}_2 is $2\sqrt{2}$ and the Bell-CHSH inequality can be violated by quantum state by a maximal factor of $\sqrt{2}$ [4].

The above brief presentation of the standard Bell-CHSH inequality, valid for 2 qubits, motivates the following generalization for n qubits which is defined recursively. Let $a_n = \pm 1$ and $a'_n = \pm 1$ denote 2 possible outcomes of 2 possible measurements on the nth qubit and define:

$$F_n \equiv \frac{1}{2}(a_n + a'_n)F_{n-1} + \frac{1}{2}(a_n - a'_n)F'_{n-1} \leq 2 \quad (4)$$

where F'_n denote the same expression F_n but with all the a_j and a'_j exchanged. The above inequality (4) holds for precisely the same reason as the standard 2 qubit Bell-CHSH inequality, as presented below eq. (1). Using again the linearity of the mean, one obtains the Bell-Klyshko inequality for n qubits:

$$E_{indep.}(F_n(a_1, \dots, a_n)) \leq 2 \quad (5)$$

Similarly to the 2 qubit case, we define the Bell operator for n qubits:

$$\mathcal{B}_n = \mathcal{B}_{n-1} \otimes \frac{1}{2}(\vec{a}_n\vec{\sigma} + \vec{a}'_n\vec{\sigma}) + \mathcal{B}'_{n-1} \otimes \frac{1}{2}(\vec{a}_n\vec{\sigma} - \vec{a}'_n\vec{\sigma}) \quad (6)$$

An upper bound on the eigenvalues of \mathcal{B}_n is given by the following lema:

$$\mathcal{B}_n^2 \leq 2^{n+1} \quad (7)$$

Again, the proof follows closely the one for the 2 qubit case:

$$\begin{aligned}\mathcal{B}_n^2 &= \mathcal{B}_{n-1}^2 \otimes \frac{1}{2}(1 + \vec{a}_n \vec{a}'_n) + (\mathcal{B}'_{n-1})^2 \otimes \frac{1}{2}(1 - \vec{a}_n \vec{a}'_n) + [\mathcal{B}_{n-1}, \mathcal{B}'_{n-1}] \otimes \frac{1}{2}i(\vec{a}_n \wedge \vec{a}'_n)\vec{\sigma} \quad (8) \\ &\leq (1 + |\vec{a}_n \wedge \vec{a}'_n|)2^n \\ &\leq 2^{n+1}\end{aligned}$$

The maximally entangled states discussed in the next section saturate the above bound. Hence, the largest eigenvalue of \mathcal{B}_n is $2^{(n+1)/2}$ and the Bell-Klyshko inequality for n qubits can be violated by quantum states by a maximal factor of $2^{(n-1)/2}$.

The maximal violation of the inequality (5) is larger than the one presented by Mermin [5] for even numbers n of qubits and is larger than the inequality derived by Ardehali [6] for odd numbers of qubits.

The inequality (5) is symmetric among the n qubits. For example, for 3 qubit it reads: $E(a, b, c') + E(a, b', c) + E(a', b, c) - E(a', b', c') \leq 2$.

An interesting property of the inequality (5) is that if one assumes that only $m < n$ qubits have independent elements of reality, then, combining the inequalities (4) and (7), one obtains:

$$E_{m \text{ indep. qubits}}(F_n) \leq 2^{(n-m+1)/2} \quad (9)$$

Indeed, applying recursively the definition (4) one obtains (see Appendix B):

$$F_n = \frac{1}{4}(F_{n-m} + F'_{n-m})F_m + \frac{1}{4}(F_{n-m} - F'_{n-m})F'_m \quad (10)$$

Hence, if the m last qubits are independent, then $F_m \leq 2$ and $F'_m \leq 2$, thus $F_n \leq F_{n-m}$. Finally, if the $n - m$ first qubits are maximally entangled, then $E_{m \text{ indep. qubits}}(F_n) = E_{QM}(F_{n-m}) \leq 2^{(n-m+1)/2}$. Note that by symmetry the same result hold if any m qubits are independent. Accordingly, from an experimental measurement result of $E(F_n)$ between $2^{(n-m+1)/2}$ and $2^{(n-m+2)/2}$ one can infer that at least m qubits are entangled [7, 8]. For example consider the following 3-qubit mixed state with only 2-qubit entanglement [9]: $\rho = \frac{1}{2}(P_S \otimes P_+ + P_+ \otimes P_S)$ where P_S denotes the singlet state of 2 qubits and $P_+ = |+\rangle_{zz}\langle +|$ the "up" state of a single qubit. Then the maximal violation of the generalized Bell-CHSH inequality (5) equals $E_{QM}(F_3) = 2(1 + \sqrt{2})$. This value is reached for $\alpha = -\alpha' = \gamma = -\gamma' = \pi/8$, $\beta = \pi$, $\beta' = \pi/2$ where α and α' are the angles defining the directions a and a' , respectively, in the xz plane (angle 0 corresponding to the z direction) and similarly for the 4 other angles. From this value one concludes rightly that the state ρ contains 2-qubit entanglement.

3 Maximally entangled states of n qubits

In this section we consider several criteria to characterize Symmetric Maximally Entangled (SME) states of n qubits.

3.1 Maximal violation of Bell inequality

Maximally entangled states should maximally violation Bell inequality. This criteria is not precise, because there are infinitely many versions of Bell inequality. In this subsection we consider the Klyshko version presented in section 2, because it is the natural generalization of the Bell-CHSH inequality which nicely characterizes the 2 qubits Bell states. The generalized Bell-Klyshko inequality (5) for n qubits is maximally violated by the GHZ [10] state $|\uparrow \dots \uparrow\rangle + |\downarrow \dots \downarrow\rangle$

for \vec{a}_j regularly distributed in the xy plane (on the Poincaré sphere) with angles $(j-1)(-1)^{n+1}\frac{\pi}{2n}$ with respect to the x-axis and $a'_j \perp a_j$. More generally, the state $\alpha|\uparrow \dots \uparrow\rangle + \beta|\downarrow \dots \downarrow\rangle$ with real α and β satisfying $\alpha^2 + \beta^2 = 1$ violate the n qubit inequality (5) by a factor $\alpha\beta 2^{(n-1)/2}$.

3.2 Maximal entanglement as maximal fragility

It seems natural to assume that entanglement is fragile. More specifically, if the qubits are subject to noise, acting independently on each of them, the more entangled states are more affected than the less entangled ones. Noise acting on a qubit can be modelled by a fluctuating Hamiltonian: $H = \vec{\beta}\vec{\sigma}$ where the 3 components of the vector $\vec{\beta}$ are independent Wiener processes (ie white noise). Accordingly, the density matrix of the n qubits follows the master equation:

$$\dot{\rho}_t = \sum_{j=1}^n (\vec{\sigma}_j \rho_t \vec{\sigma}_j - 3\rho_t) \quad (11)$$

where the Pauli matrices $\vec{\sigma}_j$ act on the jth qubit. Now, a state ψ is fragile if under the above evolution it quickly drift away, that is if $|\langle\psi|\dot{P}_\psi|\psi\rangle| = |\sum_{j=1}^n \langle\vec{\sigma}_j\rangle_\psi^2 - 3n|$ is large. Accordingly, a state ψ is maximally fragile iff $\langle\vec{\sigma}_j\rangle_\psi^2 = 0$ for all j. This condition is equivalent to the 1-qubit partial state being equal to the maximally mixed state:

$$\rho_1 \equiv Tr_{2..n}(P_\psi) = \mathbb{1}/2 \quad (12)$$

Another type of fragility, natural for entanglement, is fragile under measurement interactions: if one or some qubits are measured (in the computational basis) on a maximally entangled state, then all the entanglement is destroyed, ie the measurement projects the state onto a product state.

The GHZ states $|0\dots 0\rangle \pm |1\dots 1\rangle$ satisfy both of these 2 criteria of maximal fragility.

3.3 Distribution of entangled stated

Another criteria for maximal entanglement could be the following. A state of n qubits is maximally entangled if any k n holders of a qubit can distribute to the k-n other qubit holders a maximally entangled state. Note that this criteria applies recursively to larger and larger number of qubits, starting from the well known Bell states for 2 qubits.

At first this criteria may seem in contradiction with the one of the previous sub-section 3.2. However, this is not the case, provided the measurement is not done in the computational basis (the "z-basis"), but in the "x-basis". Indeed, using the formula of the appendix the GHZ states can be rewritten as:

$$|0, n\rangle_z + |n, n\rangle_z = (|0, k\rangle_z + |k, k\rangle_z) \otimes (|0, n-k\rangle_z + |n-k, n-k\rangle_z) + (|0, k\rangle_z - |k, k\rangle_z) \otimes (|0, n-k\rangle_z - |n-k, n-k\rangle_z) \quad (13)$$

$$= \sum_{j=0}^{k/2} |2j, k\rangle_x \otimes (|0, n-k\rangle_z + |n-k, n-k\rangle_z) + \sum_{j=0}^{(k+1)/2} |2j+1, k\rangle_x \otimes (|0, n-k\rangle_z - |n-k, n-k\rangle_z) \quad (14)$$

Hence, if k qubits are measured in the x-basis and an even number of 1 are obtained, then the remaining n-k qubits are in the GHZ state $|0, n-k\rangle_z + |n-k, n-k\rangle_z$. Else, if an odd number of 1 are obtained, the n-k qubits are in the other GHZ state $|0, n-k\rangle_z - |n-k, n-k\rangle_z$.

3.4 Mutual information of measurement outcomes on maximally entangled states

Let a_j denote the results of simultaneous measurements of all qubits in the computational basis. A natural criterion for maximal entanglement is that the mutual information of the n random variables a_j is maximum. Recall that $I(\{a_j\}) \equiv \sum_j H(a_j) - H(a_1, \dots, a_n)$, where H is the entropy function $H(x) \equiv -M(\log(p(x)))$. Using the symmetry among the a_j and the chain rule [11] $H(a_1, \dots, a_n) = H(a_1) + H(a_2, \dots, a_n|a_1)$, one obtains:

$$I(\{a_j\}) = (n-1)H(a_1) - H(a_2, \dots, a_n|a_1) \quad (15)$$

Accordingly, the mutual information $I(\{a_j\})$ is maximal if $H(a_1)$ is maximal and $H(a_2, \dots, a_n|a_1)$ is minimal. Finally, $H(a_1)$ is maximal if both outcomes are equally probable, and $H(a_2, \dots, a_n|a_1)$ is minimal if the outcome a_1 fully determines all the results a_2 to a_n . The GHZ states do clearly satisfy this criterion. Note that for $n \geq 3$ they are the only symmetric states satisfying this criterion (for $n = 2$, the third symmetric Bell state does also satisfy this criterion). Note also the similarity between this mathematical criteria and the more physical "maximally fragile" criteria of section 3.2: maximizing $H(a_1)$ is equivalent to condition (12) which represents maximal fragility under noise and minimizing $H(a_2, \dots, a_n|a_1)$ is equivalent maximal fragility under 1-qubit measurement in the computational basis.

3.5 Partial states of maximally entangle states are maximally mixed?

In subsection 3.2 we found that the 1-qubit partial state of a maximally fragile states is the maximally mixed state. A more general criterion for maximal entanglement could be that all partial states are maximally mixed. Since we assume all through this article symmetric states, maximally mixed states differ from states represented by a multiple of the identity matrix: maximally mixed states are homogenous mixtures of all the $n+1$ symmetric states. For example, the maximally mixed state of 2-qubit is the mixture of the 3 symmetric Bell states, all 3 with the same weight. Note that the density matrices representing m -qubits and $(n-m)$ -qubits partial states of an n -qubit state have the same spectra. Hence, only the partial states for $m = 1, \dots, [\frac{n}{2}]$ can be maximally mixed (where $[\frac{n}{2}]$ is the largest integer smaller or equal to $\frac{n}{2}$). Consequently, the m -qubit partial states of a SME state should, according to this criteria, have exactly $m+1$ identical eigenvalues different from zero, for all $m = 1, \dots, [\frac{n}{2}]$. Note that the conditions for different m 's are not independent. Indeed, if the partial state for $m = [\frac{n}{2}]$ is maximally mixed, then all partial states for smaller m 's are also necessarily maximally mixed.

We found states of n -qubits satisfying this criterion for $n=2,3,4$ and 6. However, no such state exists for $n=5$, nor for $n \geq 7$! Hence, in general, no state satisfying this criterion exists. This is no surprise, since the number of constraints increases with n much faster than the number of parameters defining symmetric states. This criterion is thus not suitable to characterize maximal entanglement. Indeed, if entanglement can be measured, then maximally entangled states should exist for any number of qubits.

For completeness we nevertheless list some examples of states satisfying this criterion:

$$\psi_{3,\pm 1} = |0, 3\rangle \pm |3, 3\rangle \quad (16)$$

$$\psi_{3,\pm 2} = |0, 3\rangle \pm |1, 3\rangle - |2, 3\rangle \mp |3, 3\rangle \quad (17)$$

$$\psi_{4,\pm 1} = -3|0, 4\rangle \pm \sqrt{3}|1, 4\rangle + |2, 4\rangle \pm \sqrt{3}|3, 4\rangle - 3|4, 4\rangle \quad (18)$$

$$\psi_{4,2} = |0, 4\rangle + \frac{i}{\sqrt{3}}|2, 4\rangle + |4, 4\rangle \quad (19)$$

$$\psi_{6,\pm 1} = |1, 6\rangle \pm |5, 6\rangle \quad (20)$$

$$\psi_{6,2} = -3|0, 6\rangle + |2, 6\rangle + |4, 6\rangle - 3|6, 6\rangle \quad (21)$$

$$\psi_{6,\pm 3} = \sqrt{2}|0, 6\rangle \pm \frac{i}{2}|3, 6\rangle + \sqrt{2}|6, 6\rangle \quad (22)$$

where $|j, n\rangle$ denotes the sum of all product states with j $|1\rangle$ and $n - j$ $|0\rangle$ (see Appendix A) and where the first index of the ψ 's indicate the number of qubits and the second index label the different states with maximally mixed partial states.

4 Conclusion

The Bell-Klyshko inequality for n qubits has been presented. Maximally entangled quantum states violate this inequality by a factor that grows exponentially with n . For even and odd numbers of qubits, this maximal violation is larger than for the inequality devised by Mermin [5] and by Ardehali [6], respectively. If only $n - m$ qubits are assumed independent, then the same inequality leads to a higher bound. Hence, from an experimental test of the Bell-Klyshko inequality for n qubits, one can infer a lower bound on the number of entangled qubits.

Maximally entangled symmetric states of n qubits were analysed according to 5 different criteria. The criteria that the partial states should be maximally mixed is found to be of limited value, since no such states exist neither for $n = 5$ nor for $n \geq 7$. However, all 4 other criteria are shown to be compatible. The conclusion is that the 2 GHZ states [10] are the maximally entangled symmetric states: they violate Bell inequality maximally, they are maximally fragile and maximize the mutual information. Moreover, depending on the measurement bases, m holders of qubits can either distribute maximally entangled qubits to their $n - m$ colleagues or leave them with product states.

Clearly all states obtained by local transformation of a maximally entangled state are equally valid maximally states [12], though not necessarily symmetric states. Starting from the GHZ state one thus obtains 2^n linearly independent maximally entangled states. Hence, the maximally entangled states form an orthogonal basis for any number of qubits, like the well known Bell states for 2 qubits. For example, in the 3 qubit case the two GHZ states read $|1, 1, 1\rangle \pm |0, 0, 0\rangle$. By local transformation on the first qubit the following pair of states obtains: $|0, 1, 1\rangle \pm |1, 0, 0\rangle$. Acting similarly on the second and third qubits provides a total of 8 mutually orthogonal states that form a basis of the 3-qubit Hilbert space.

In this letter we did not address the question of unicity. However, our experience leads us to conjecture that the GHZ states and the states obtained from them by local transformations are the unique states that violate maximally the Bell inequality (5) and are the unique states satisfying independently the criteria of sub-sections 3.2, 3.3 and 3.4. Hence, we conjecture that our 4 criteria are not only compatible, but equivalent characterization of maximally entangled states.

Acknowledgments

Stimulating discussions with A. Ekert, B. Huttner, S. Popescu, A. Zeilinger and M. Żukowski are acknowledged. In particular we thank M. Żukowski for bringing references [3] to our attention. H.B.-P. is supported by the Danish National Science Research Council (grant no. 9601645). This work profited also from support by the Swiss National Science Foundation.

Appendix A: symmetric states of n qubits

In this appendix we summarize some useful properties and notations for symmetric states. Let $|j, n\rangle$ denote the sum of all product states with j $|1\rangle$ and $n-j$ $|0\rangle$. For example: $|1, 3\rangle \equiv |1, 0, 0\rangle + |0, 1, 0\rangle + |0, 0, 1\rangle$. Hence $|j, n\rangle$ is a 2^n component vector representing a n-qubit state. The norms and inner products are given by:

$$\langle j, n | k, n \rangle = \delta_{j,k} C_j^n \equiv \delta_{j,k} \frac{n!}{(n-j)!j!} \quad (23)$$

The decomposition of symmetric states of n qubits on symmetric states of m qubits states is straightforward:

$$|j, n\rangle = \sum_{k=0}^j |k, m\rangle \otimes |j-k, n-m\rangle \quad (24)$$

It is also relatively easy to change from one basis to another. For example, if $|1\rangle_x \equiv |0\rangle_z + |1\rangle_z$ and $|0\rangle_x \equiv |0\rangle_z - |1\rangle_z$, then:

$$|j, n\rangle_z = \sum_{\ell=0}^n \left(\sum_{k=0}^{\frac{j-\ell}{2}} C_{j-2k}^\ell C_{2k}^{n-\ell} - \sum_{k=0}^{\frac{j-\ell-1}{2}} C_{j-2k-1}^\ell C_{2k+1}^{n-\ell} \right) |\ell, n\rangle_x \quad (25)$$

Accordingly the GHZ states expressed in the x -bases read:

$$|0, n\rangle_z + |n, n\rangle_z = \sum_{k=0}^{n/2} |2k, n\rangle_x \quad (26)$$

$$|0, n\rangle_z - |n, n\rangle_z = \sum_{k=0}^{(n+1)/2} |2k+1, n\rangle_x \quad (27)$$

Similarly, the GHZ states expressed in y -basis ($|1\rangle_y \equiv |1\rangle_z + i|0\rangle_z$, $|0\rangle_y \equiv |0\rangle_z + i|1\rangle_z$) read:

$$|0, n\rangle_z \pm |n, n\rangle_z = \sum_{k=0}^n (i^k \pm i^{n-k}) |k, n\rangle_y \quad (28)$$

Appendix B: Proof of formula (10)

Let us prove it by induction on m . Defining $F_1(a) \equiv 2a$, formula (10) holds for $m = 1$ and, recalling that $F_m'' = F_m$, one has:

$$F_n = \frac{F_m + F_m'}{4} F_{n-m} + \frac{F_m - F_m'}{4} F_{n-m}' \quad (29)$$

$$= \frac{F_m + F_m'}{4} \left(\frac{a_{m+1} + a_{m+1}'}{2} F_{n-(m+1)} + \frac{a_{m+1} - a_{m+1}'}{2} F_{n-(m+1)}' \right) \quad (30)$$

$$+ \frac{F_m - F_m'}{4} \left(\frac{a_{m+1}' + a_{m+1}}{2} F_{n-(m+1)}' + \frac{a_{m+1}' - a_{m+1}}{2} F_{n-(m+1)} \right)$$

$$= \frac{F_m a_{m+1}' + F_m' a_{m+1}}{4} F_{n-(m+1)} + \frac{F_m a_{m+1} - F_m' a_{m+1}'}{4} F_{n-(m+1)}' \quad (31)$$

$$= \frac{F_{m+1} + F_{m+1}'}{4} F_{n-(m+1)} + \frac{F_{m+1} - F_{m+1}'}{4} F_{n-(m+1)}' \quad (32)$$

Hence, assuming (10) holds for m , it also holds for $m+1$.

References

- [1] J.S. Bell, *Physics* **1**, 195, 1964.
- [2] J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, *Phys. Rev. Lett.* **23**, 880, 1969.
- [3] D.N. Klyshko, *Phys. Lett. A* **172**, 399, 1993; A.V. Belinskii and D.N. Klyshko, *Physics - Uspekhi* **36**, 653, 1993.
- [4] B.S. Cirel'son, *Lett. Math. Phys.* **4**, 93, 1980.
- [5] N.D. Mermin, *Phys. Rev. Lett.* **65**, 1838, 1990.
- [6] M. Ardehali, *Phys. Rev. A* **46**, 5375, 1992.
- [7] G. Svetlichny, *Phys. Rev. D* **35**, 3066, 1987.
- [8] M. Zukowsky and D. Kaszlikowski, *Phys. Rev. A* **56**, R1682, 1997.
- [9] G. Brassard and T. Mor, *Multi-particle entanglement via 2-particle entanglement*, preprint, Université de Montréal, 1998.
- [10] D.M. Greenberger, M. Horne and A. Zeilinger in *Bell's theorem, Quantum theory and conceptions of the universe*, ed. M. Kafatos, Kluwer 1989.
- [11] Th. M. Cover and J.A. Thomas, *Elements of information theory*, ed. John-Wiley and Sons, 1991.
- [12] N. Linden and S. Popescu, quant-ph 9711016.